Announcements

1) HW 1 up, due 1/23

2) Subs for this week!

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Warning:

The book repeatedly refers to terms and results later in the text.

Preliminaries

Notation: $(\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}, \mathbb{N})$

$\mathbb{R}$ = the real numbers

$\mathbb{C}$ = the complex numbers

$\mathbb{Q}$ = the rational numbers

$\mathbb{Z}$ = the integers

$\mathbb{N}$ = the natural numbers (excluding zero!)

Finite-dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$ and the linear maps between them

Every finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ is isomorphic as a vector space to $\mathbb{R}^n$ or $\mathbb{C}^n$, where n is the real or complex dimension, respectively.

Mathematical Shorthand

"$\exists$" = there exists

"$\forall$" = for every

"$\Rightarrow$" = implies

"$\in$" = in

"$\subseteq$" = is contained in

More notation

"$\{ \qquad \}$"

means "the set of".

Notation like

$$\{x \in \mathbb{C}^2 \mid x = (x_1, x_2),\ x_1 = 0\}$$

reads "the set of all x in $\mathbb{C}^2$ such that $x = (x_1, x_2)$ and $x_1 = 0$."

Complex Numbers: why?

If p is a polynomial with real (or complex) coefficients and degree n.

Then the Fundamental Theorem of Algebra guarantees us that p has n roots, counted with multiplicity

PROVIDED we allow the roots to be complex numbers.

Representation:

Denote by i the square root of -1. Every complex number has a representation as

$$z = x + iy$$

for x, y real numbers.

So a complex number

= an ordered pair

of real numbers.

We can just think

of complex numbers

as elements of $\mathbb{R}^2$,

with $(1,0)$ thought of as

the number 1 and $(0,1)$

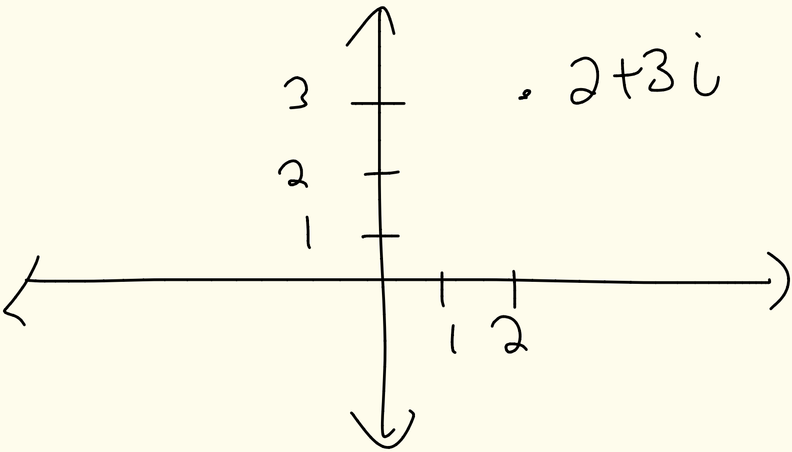
thought of as i.

Picture

Plot $2+3i$

$= (2,0) + (0,3)$

$= (2,3)$



Complex Conjugate and Absolute Value

If $z = x + iy$ with x, y real, the complex conjugate of z is denoted by

$$\boxed{\bar{z} = x - iy}$$

The absolute value (or modulus) of a complex number $z = x + iy$ is given by

$$\boxed{|z| = (z\bar{z})^{1/2}}$$

Example 1:

$$z = 5 - 4i$$

$$\bar{z} = 5 + 4i$$

$$|z| = (z\bar{z})^{1/2}$$

$$= ((5+4i)(5-4i))^{1/2}$$

$$= (25 + \cancel{20i} - \cancel{20i} + 16)^{1/2}$$

$$= \sqrt{41}$$

Matlab Calling Command

abs(z) gives

the absolute value

of a vector z.

Vandermonde Matrix

Given $n \in \mathbb{N}$ and $(x_i, y_i) \in \mathbb{R}^2$, $1 \leq i \leq n$, $x_i \neq x_j$ for $i \neq j$, $\exists$ a unique polynomial p of degree $n-1$ with real coefficients satisfying

$$\boxed{p(x_i) = y_i}$$

for $1 \leq i \leq n$.

p is said to interpolate $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$

$n=2$: through any two distinct points, there is a unique line.

$n>2$ - not so easy, even to visualize how it would work!

Linear Algebra Formulation

$n=2$

Given points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$, we want a line

$y = c_1 x + c_0$ such that

$$y_1 = c_1 x_1 + c_0$$
$$y_2 = c_1 x_2 + c_0$$

Rewrite as a vector equation

$$\begin{bmatrix} c_1 x_1 + c_0 \\ c_1 x_2 + c_0 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

then as a matrix equation

$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}}_{\text{2x2 Vandermonde Matrix}} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$n > 2$

Given $\{(x_i, y_i)\}_{i=1}^{n}$

with $x_i \neq x_j$ for $i \neq j$,

we find a polynomial

$$p(x) = \sum_{k=0}^{n-1} c_k x^k$$ and

$$p(x_i) = y_i \text{ for all } 1 \leq i \leq n$$

by

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}}_{\substack{n \times n \\ \text{Vandermonde} \\ \text{Matrix}}} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Definition: (Vandermonde matrix)

The $n \times n$ Vandermonde matrix is the matrix

$$A_n = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

for data $(x_1, x_2, \ldots, x_n)$.

Solution: A_n invertible $\forall n$.

why?

First, notation: ($\mathbb{C}^{m\times n}$)

An $m\times n$ matrix is written

as an element of $\mathbb{C}^{m\times n}$

in the text. If $m=n$,

I will sometimes use $M_n(\mathbb{C})$.

Book Convention

Usually $m \geq n$.

Theorem: (invertibility equivalences)

Let $A \in \mathbb{C}^{n \times n}$. Then the following are equivalent

1) A is invertible

2) $\det(A) \neq 0$

3) $\operatorname{rank}(A) = n$

4) $\operatorname{nullity}(A) = 0$

5) $\dim(\text{range}(A)) = n$

$$\boxed{\text{6) } \ker(A) = \text{null}(A) = \{\vec{0}\}.}$$

7) The columns of A are linearly independent

8) 0 is not an eigenvalue of A

9) 0 is not a singular value of A (later!)

Recall (injective/surjective)

If $A \in \mathbb{C}^{n \times n}$, then

A is injective if and only if A is surjective.

surjective: $\operatorname{rank}(A) = n$

injective: $\ker(A) = \{\vec{0}\}$

Back to Vandermonde Matrix

Show $\ker(A_n) = \{\vec{0}\}$.

Suppose there is a vector $c = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$

with $A_n \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Unravelling,

$$\sum_{k=0}^{n-1} c_k x_i^k = 0$$

for all $1 \leq i \leq n$.

If $x_i \neq x_j$, this

says the polynomial

$$p(x) = \sum_{k=0}^{n-1} c_k x_i^k$$ has

n zeros!

This violates the fundamental theorem of algebra since that is one too many roots — unless $c_0 = c_1 = \cdots = c_{n-1} = 0$ (p is the zero polynomial).

That proves A_n has only $\{\vec{0}\}$ in its kernel.

Example 2

Consider $(1,2)$, $(3,15)$, $(7,-1)$. We want a quadratic that interpolates the x & y coordinates. This is solving

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 7 & 49 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ -1 \end{bmatrix}$$

Use Matlab's

$A \backslash b$ to solve

for c_0, c_1, c_2

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 7 & 49 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 15 \\ -1 \end{bmatrix}$$

$c_0 = -9.75, \ c_1 = 13.5$

$c_2 = -1.75$